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On Defect Groups and p-Constraint

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Theorem 1 of [7] asserts that if D is a nontrivial defect group for a non-principal p-block of the finite group, G, and H is a p-constrained subgroup of G containing $C_G(D)$, then $O_{p'}(H) \neq \langle 1 \rangle$. We shall construct counter-examples to this for all primes p. In our counterexample for the prime 2, H will in fact be a 2-constrained 2-local subgroup of G. We shall obtain, as well, some partial results which are sufficient for the proofs of Corollaries 2–5 in [7] and for all of the applications in [8]. Thanks are due to Leonard Scott, David Goldschmidt and Jonathan Alperin for several valuable observations.

For our examples and for our positive results, we need certain well known facts which we collect in the following lemmas. For completeness, we include proofs of these.

LEMMA 1. *Let H be a p-constrained group, D a p-subgroup of H, and M a subgroup of H with $DC_H(D) \subseteq M \subseteq N_H(D)$. Then M is p-constrained and $O_{p'}(M) \subseteq O_{p'}(H)$.*

Proof. We may assume that $O_{p'}(H) = \langle 1 \rangle$. Let $P = O_p(M)$ and let Q be a P-invariant p'-subgroup of M. Then $P \times Q$ acts on $B = O_p(H)$. As $D \subseteq P$, we have

$$C_H(P) \subseteq C_H(D) \subseteq M \subseteq N_H(P).$$

Thus, as $C_B(P) \trianglelefteq C_H(P)$, we have

$$C_B(P) \subseteq O_p(C_H(P)) \text{ char } C_H(P) \trianglelefteq M.$$

Hence, $C_B(P) \subseteq O_p(M) = P$. Thus $[C_B(P), Q] = \langle 1 \rangle$, whence $[B, Q] = \langle 1 \rangle$ by the $P \times Q$-lemma. Hence, $Q = \langle 1 \rangle$. Thus, $O_{p'}(M) = \langle 1 \rangle$ and $C_M(P)$ is a normal p-subgroup of M. Hence $C_M(P) \subseteq P$ and M is p-constrained.

COROLLARY. *Let D be a p-subgroup of a finite group, G, with $C_G(D)$ p-constrained. Let P be a p-subgroup of G containing D. Then $C_G(P)$ is p-constrained.*

Proof. We proceed by induction on the index of D in P. As D is subnormal in P, we may assume that D is normal in P. Thus $PC_G(P) \subseteq N_G(D)$. As $C_G(D)$ is p-constrained, so is $N_G(D)$ and Lemma 1 applies.

LEMMA 2. *Let D be a nontrivial p-subgroup of the finite group, G. Suppose that $C_G(D)$ is p-constrained.*

(i) *If $O_{p'}(C_G(D)) = \langle 1 \rangle$, then $DC_G(D)$ has no nonprincipal p-block.*

(ii) *If D is a defect group for a nonprincipal p-block of G, then $O_{p'}(C_G(D)) \neq \langle 1 \rangle$.*

(iii) *Suppose that $O_{p'}(C_G(D)) \neq \langle 1 \rangle$ and that $O_{p'}(C_G(P)) = \langle 1 \rangle$ for all p-subgroups, P, of G properly containing D. Then D is a defect group for a nonprincipal p-block of G.*

Proof. (i) Suppose that $O_{p'}(C_G(D)) = \langle 1 \rangle$. Suppose that P is a defect group for a nonprincipal p-block of $DC_G(D)$. By [1, 9F], $O_p(DC_G(D)) \subseteq P$. Thus $C_G(P) \subseteq P$. But then $P = PC_G(P)$ has no nonprincipal p-block. Hence, $DC_G(D)$ has no nonprincipal block with defect group, P, by [2, Theorem 3], a contradiction.

(ii) Suppose that $O_{p'}(C_G(D)) = \langle 1 \rangle$. Then, by (i), $DC_G(D)$ has no nonprincipal p-block. Thus, by [3, Theorem 5C] and [2, Theorem 3], G has no nonprincipal p-block with defect group, D.

(iii) Suppose that D is maximal with respect to the property that $O_{p'}(C_G(D)) \neq \langle 1 \rangle$. It follows from [2, Theorem 1] and [1, 9F] that $N = N_G(D)$ has a nonprincipal p-block with defect group, P, containing D. Now N is p-constrained and $PC_G(P) \subseteq N$. Thus, by Lemma 1, $PC_G(P)$ is p-constrained, whence $N_N(P)$ is p-constrained. By [3, 5C] and [2, Theorem 3], $N_N(P)$ has a nonprincipal p-block with defect group, P. Thus by part (i), $O_{p'}(C_N(P)) = O_{p'}(C_G(P)) \neq \langle 1 \rangle$. Hence, by hypothesis, $P = D$. Then by [3, 5C] and [2, Theorem 3], G has a nonprincipal p-block with defect group D.

We remark that Lemma 2 is proved in [8] under the hypotheses that all p-local subgroups of G are p-constrained. The next result may be found in [4].

LEMMA 3. *Let D be a defect group for the p-block, B, of G. Let D_1 be a subgroup of D, and H a subgroup of $N_G(D_1)$ containing $D_1C_G(D_1)$. Then there exists a p-block, b, of H with $b^G = B$.*

Proof. Let F be a p-adic number field which is a splitting field for G and let $\bar{F}$ be the corresponding residue class field. Let E_B be the primitive idempotent of $Z(\bar{F}[G])$ corresponding to the block, B. Write $E_B = \sum_i a_i K_i$ where $a_i \neq 0$ and $K_i = \sum_{x \in C_i} x$ for the class, C_i, of G. For some j, K_j has defect group, D. Let $S: Z(\bar{F}[G]) \to Z(\bar{F}[H])$ be the Brauer homomorphism. As $C_G(D_1) \supseteq C_G(D)$, we see that $K_j \cap C_G(D_1)$ is not empty. Thus $S(K_j) \neq 0$, whence $S(E_B) \neq 0$. Writing $S(E_B) = \sum E_i$, where E_i is the primitive idempotent of $Z(\bar{F}[H])$ corresponding to the block, b_i, of H, we have $b_i^G = B$.

Our counterexample for the prime 2 is a central extension, G, of Z_2 by S_7, the symmetric group of degree 7. Specifically, there is a unique group, G, constructed by Schur [5] such that $Z(G) = Z(G) \cap G' = \langle z \rangle \cong Z_2$; such that $\bar{G} = G/Z(G) \cong S_7$; and such that if $g \in G$ with $\bar{g} = (1\ 2) \in \bar{G}$, then $g^2 = 1$. We shall use bars to denote the homomorphic images of elements and subgroups of G in $\bar{G}$. According to Schur, if $\bar{x} \in \bar{G}$ is a transposition, then $x^2 = 1$. If $\bar{y} \in \bar{G}$ is a product of two or three transpositions, then $y^2 = z$. Let D be the full inverse image of $\bar{D} = \langle (1\ 2), (1\ 3)(2\ 4) \rangle$ in G. Then D is quasidihedral with center, $\langle z \rangle$. Let $r \in G$ of order 3 with $\bar{r} = (5\ 6\ 7)$ and let $t \in G$ with $\bar{t} = (5\ 6)$. Then $N_G(D) = \langle D, r, t \rangle$ and $C_{\bar{G}}(\bar{D}) = \langle (1\ 2)(3\ 4), (5\ 6\ 7), (5\ 6) \rangle$. Pick elements a, b and c in G with $\bar{a} = (1\ 2)$, $\bar{b} = (1\ 3)(2\ 4)$ and $\bar{c} = (1\ 2)(3\ 4)$. If $g \in C_G(D)$, then $g = c^i t^j r^k$ and $c^i t^j \in C_G(D)$. Now $(ac)^2 = 1$ and $a^2c^2 = z$. Thus $c \notin C_G(D)$. Also $(at)^2 = z \neq a^2t^2$ and $(bct)^2 = z \neq b^2(ct)^2$. Thus $t \notin C_G(D)$ and $ct \notin C_G(D)$. Hence, $C_G(D) = \langle z, r \rangle$. Now $O_{2'}(C_G(D)) \neq \langle 1 \rangle$ and if P is a 2-subgroup of G properly containing D, then $P \in \mathrm{Syl}_2(G)$ and $C_G(P) = Z(P)$. Thus by Lemma 2, D is a defect group for a nonprincipal 2-block of G. Let $\bar{A} = \langle (1\ 2), (4\ 5)(6\ 7), (4\ 6)(5\ 7) \rangle$. Then $N_{\bar{G}}(\bar{A}) = \langle \bar{A}, \bar{r}, \bar{t} \rangle$. Let A be the full inverse image of $\bar{A}$ in G. Then $H = N_G(A) = \langle A, r, t \rangle$. Thus H is a 2-constrained 2-local subgroup of G; $C_G(D) \subseteq H$; and $O_{2'}(H) = \langle 1 \rangle$. Thus (G, D, H) as above affords a counterexample to Theorem 1 of [7] with H a 2-constrained 2-local subgroup of G. We remark that if G_0 is the other nonsplit central extension of Z_2 by S_7, then G_0 affords a counterexample in the same way.

Let p be an odd prime. By Dirichlet's theorem, there exists a prime q, such that p divides $q - 1$ but p^2 does not divide $q - 1$. Our counterexample for the prime p is a split extension, $G = G_0 \cdot \langle x \rangle$, where $G_0 = SL(p, q^2)$ and x is a field automorphism of order 2. Let α be a primitive pth root of unity in $GF(q)$. We pick elements

$$a = \begin{pmatrix} \alpha & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}$$

in $GL(p, q)$. Then $\langle a, b\rangle \simeq Z_p \wr Z_p$ is a p-subgroup of $GL(p, q) \subseteq GL(p, q^2)$ Let A be the maximal abelian subgroup of $\langle a, b\rangle$ and let $N = N_{GL(p,q^2)}(A)$. Then $N = H \cdot S$ where H is isomorphic to a direct product of p copies of $GF(q^2)^\times$, $S \subseteq SL(p, q)$, and $S \simeq S_p$, the symmetric group of degree p. As A char $\langle a, b\rangle$ and $\langle a, b\rangle \in \mathrm{Syl}_p(N)$, we conclude that $\langle a, b\rangle \in \mathrm{Syl}_p(GL(p, q^2))$ and $C_{GL(p,q^2)}(\langle a, b\rangle) = C_H(b) = Z(GL(p, q^2))$. Also, $N_{GL(p,q^2)}(\langle a, b\rangle) \supseteq \langle a, b, t\rangle$, where $\langle b, t\rangle$ is a dihedral subgroup of S of order $2p$. Let $D = \langle a, b\rangle \cap G_0$ and let $N_0 = \langle a, b, t\rangle \cap G_0$. Then $D \in \mathrm{Syl}_p(G)$ and $D \subseteq C_G(x)$. Moreover, $C_G(D) = Z(D) \times \langle x\rangle$ and $t \in N_0 \subseteq N_G(D)$. Hence by Lemma 2, D is a defect group for a nonprincipal p-block of G. Now xt is an involution of $G - G_0$. Thus, by Lang's theorem, (c.f. [6, Section 10]), xt is G-conjugate to x. As $xt \in N_G(D) - C_G(D)$, there exists a Sylow p-subgroup, D_1, of G with $x \in N_G(D_1) - C_G(D_1)$. Let $H = \langle D_1, x\rangle$. Then H is a solvable subgroup of G containing $C_G(D)$ and $O_{p'}(H) = \langle 1\rangle$. Thus (G, D, H) is a counterexample for the prime p. We remark that an analogous counterexample for the prime 2 is afforded by the split extension, $G = G_1 \cdot \langle y\rangle$, where $G_1 = SL(2, 3^3)$ and y is a field automorphism of order 3. We also note that these groups also afford counterexamples to Corollary 1 of [7].

We now prove our positive results.

THEOREM 1. *Let D be a p-subgroup of a finite group G, with $O_{p'}(C_G(D)) \neq \langle 1\rangle$. Let H be a p-constrained subgroup of G containing $C_G(D)$. If $C_G(Z(D))$ is p-constrained, then $O_{p'}(H) \neq \langle 1\rangle$.*

Proof. As $DC_G(D) \subseteq C_G(Z(D))$, $O_{p'}(C_G(D)) \subseteq O_{p'}(C_G(Z(D)))$ by Lemma 1. Thus $O_{p'}(C_H(Z(D))) \neq \langle 1\rangle$. Thus $O_{p'}(H) \neq \langle 1\rangle$ by Lemma 1, since $C_H(Z(D)) = Z(D)\, C_H(Z(D))$.

COROLLARY. *Let D be a defect group for a nonprincipal p-block of the finite group, G. Let H be a p-constrained subgroup of G containing $C_G(D)$. If $C_G(Z(D))$ is p-constrained, then $O_{p'}(H) \neq \langle 1\rangle$.*

Proof. By Lemma 2(ii), $O_{p'}(C_G(D)) \neq \langle 1\rangle$. Thus Theorem 1 applies.

We remark that the above corollary suffices for the proofs of Corollaries 3–5 of [7] and for all of the applications in [8].

THEOREM 2. *Let D be a defect group for a nonprincipal p-block of the finite group, G. Let H be a p-constrained subgroup of G containing $C_G(D \cap H)$. Then $O_{p'}(H) \neq \langle 1\rangle$.*

Proof. Suppose that $O_{p'}(H) = \langle 1\rangle$. Let $H_0 = (D \cap H)\, C_G(D \cap H) \subseteq H$. By Lemma 3, there is a p-block, b, of H_0 with $b^G = B$. Then, by [2, Theorem 3], b is a nonprincipal block of H_0. By Lemma 1, H_0 is p-constrained and

$O_{p'}(H_0) = \langle 1 \rangle$. But then, by Lemma 2(i), H_0 has no nonprincipal p-block, a contradiction.

COROLLARY. *Let D be a defect group for a nonprincipal p-block of the finite group, G. Let D_1 be a subgroup of D. Let H be a subgroup of G containing $D_1C_G(D_1)$. If H is p-constrained, then $O_{p'}(H) \neq \langle 1 \rangle$.*

Proof. As $D_1 \subseteq D \cap H$, $C_G(D \cap H) \subseteq C_G(D_1) \subseteq H$. Thus, Theorem 2 applies.

We note that Corollary 2 of [7] is immediate from the above corollary.

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